Lecture notes on risk management, public policy, and the financial system Volatility behavior and forecasting

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Volatility forecasting

Time variation in return volatility and correlation

Time variation in return volatility Time variation in return correlation

Volatility forecasting

L Time variation in return volatility

Volatility forecasts

- A major departure from standard model: *risk* or *volatility changes over time*
- Volatility, unlike return, not directly observable, must be estimated
 - Challenge: method for estimating volatility that captures typical patterns of volatility
- Recent past and long-term volatility help predict future volatility
 - But: while estimators efficacious for forecasting near-term volatility, they often miss sharp changes in volatility
- Second-moment efficiency: option market does less-poor job forecasting return variance than forward markets of forecasting mean return

Time variation in return volatility

Typical patterns of volatility behavior over time

Persistence: volatility tends to stay near its current level

- · Periods of high or low volatility tend to be enduring
- Once a large-magnitude return shocks volatility higher, volatility persists at its higher level
- Magnitude or square of return as well as return volatility display positive **autocorrelation**

Abrupt changes in volatility are not unusual

- Together with persistence, leads to volatility clustering or volatility regimes
- Shifts from low to high volatility are more abrupt, while shifts from high to low volatility are more gradual

Long-term mean reversion: volatility of an asset's return tends to gravitate to a long-term level

• In turn implies a **term structure of volatility:** different current estimates of volatility for different time horizons

— Time variation in return volatility

Volatility of oil prices 1986-2018



Prices and returns of Cushing, OK West Texas Intermediate (WTI) crude oil. 99.9 percent confidence interval calculated using daily EWMA volatility estimate (decay factor $\lambda = 0.94$). Daily, 02Jan1986–16Jan2018. *Data source*: U.S. Energy Information Administration (https://www.eia.gov/dnav/pet/pet_pri_spt_s1_d.htm).

Time variation in return volatility

Conditional volatility

- Volatility regimes suggest use of **conditional volatility**: estimate weighted toward more recent information
- Formally, volatility forecasts based on some information ("shocks" or "innovations") up to present time t
 - $\sigma_t \equiv$ current estimate of future return volatility based on (a model and) information through time t
- What new information drives σ_t ? In most models:
 - Magnitude (and possibly the sign) of recent returns
 - Recent estimates of *volatility*
- Term structure of volatility, e.g. weekly volatility higher or lower than daily
 - Typically, volatility expected to rise (fall) when low (high) relative to long-term average level

Time variation in return correlation

Impact of time-varying correlation

- Like volatility, correlations vary over time
- Correlations have strong impact on portfolio returns, hedged positions
- Abrupt changes in correlation during periods of financial stress \rightarrow
 - Failure of hedging strategies
 - Diminution of diversification benefits
- "Risk-on risk-off" behavior: tendency for correlations across many assets to rise when risk appetites diminish in stress periods
- Examples:
 - Increase in return correlations among equities
 - · Higher correlation between equity returns, Treasury yields

Time variation in return volatility and correlation

Time variation in return correlation

Time-varying correlation of stock and bond returns

- Persistent changes over time in general level of correlation between of stock and bond returns
 - 1960s-1990s: generally negative
 - 1990s-: generally positive
- Experience during the inflation and disinflation from late 1960s
 - Rising rates driven by rising inflation expectations, associated with adverse impact on economic growth
 - "Fed model": increase in discount rate for future earnings reduces present value
- Experience once low-inflation monetary policy fully credible
 - Rising rates driven by increases in anticipated real returns (*r**), associated with poitive impact on economic growth
 - Risk-on risk-off: investors reduce allocations to risky in favor of safe assets
- N.B. positive correlation of equity returns and yield changes corresponds to negative correlation of stock and bond returns

- Time variation in return volatility and correlation
 - Time variation in return correlation

Correlation of stock returns and rates 1962-2024



UST-SPX correlation — Core CPI inflation Upper panel: constant-maturity yield of 10-year U.S. Treasury note, year-over-year CPI-U inflation, log of S&P 500 index. Lower panel: correlation between log S&P price return and changes in 10-year yield (estimated via EWMA with decay factor $\lambda = 0.97$). Vertical shading represents NBER recession dates. Weekly data, 05Jan1962 to 17May2024. Data source: Bloomberg L.P.

Volatility forecasting

Simple approaches conditional volatility estimation GARCH The exponentially-weighted moving average model

Volatility forecasting

Simple approaches conditional volatility estimation

Using conditional volatility estimators

- General approach: revise most recent estimate of volatility based on most recent return data
- Simplified notation when working with daily data: return from yesterday's to today's close

$$r_t \equiv r_{t-1,t} \equiv \ln(S_t) - \ln(S_{t-1})$$

- At close of each day t, use r_t to update yesterday's volatility estimate σ_{t-1}
- Use the new estimate σ_t to measure risk or forecast volatility over the next business day t+1
- Volatility forecast horizon includes trading days, not calendar time
 - Price can change only when market open
 - Holding period and cash flows accrue every calendar day
- Choice of return **frequency** used in estimates: weekly, daily, intra-day, etc.
 - Returns independent⇒return frequency doesn't affect volatility estimates over long periods
- Volatility typically calculated using daily close-to-close returns

Simple approaches conditional volatility estimation

Volatility is easier to estimate than mean

- Imagine asset return approximately follows diffusion with drift
 - Observed at regular intervals over a period of time
 - Drift and volatility may change over time, but slowly
- You only observe one sample path in real history
- The only information on mean/drift is return over entire period
- But finer intervals—every 5 min. instead of daily—provide more information on volatility
 - Finer intervals provide more information on tendency to wander
 - Confidence interval of volatility estimate $\rightarrow 0$
 - But not confidence interval of mean estimate
- Tail risk very hard to estimate

Simple approaches conditional volatility estimation

Zero-mean assumption

- A typical risk-measurement modeling choice:
 - Estimate return volatility
 - But assume mean return = 0
- I.e. computed as **root mean square** rather than standard deviation of returns over sample period
- In lognormal model, assume drift $\mu = 0 \Rightarrow$:
 - Mean logarithmic return $\mu = 0$

$$r_{t,t+ au} = \ln(S_{t+ au}) - \ln(S_t) \sim \mathcal{N}(0,\sigma^2 au)$$

• But discrete returns have non-zero mean due to Jensen's Inequality term:

$$\mathsf{E}\left[S_{t+\tau}\right] = S_t e^{\frac{1}{2}\sigma^2 \tau}$$

Simple approaches conditional volatility estimation

Why assume zero-mean returns?

- Because we can:
 - Small impact of mean on volatility over short intervals
 - But: mean return increases linearly with time, return volatility increases as square root of time
 - \Rightarrow Over longer periods, mean has larger impact than volatility
- Because we must:
 - Expected return very hard to measure
 - Estimation of mean introduces additional source of statistical error into variance estimate
 - Bad enough to assume return normality, let's not also assume a value for expected return
 - Consider a sequence of large price changes in the same direction:
 - Incorporating the mean return would misleadingly reduce the measured volatility
 - (\rightarrow) Autocorrelation of returns

Simple approaches conditional volatility estimation

Square-root-of-time rule

- In standard (→)geometric Brownian motion model, variance (vol squared) of price change proportional to time elapsed
 - Position after t time units

$$S_t \sim \mathcal{N}(0, t)$$

• Together with martingale property

$$S_{t+\tau} - S_t \sim \mathcal{N}(0, \tau)$$

- Carries over to standard lognormal/geometric Brownian motion model: variance increases in proportion to time elapsed
 - \Rightarrow Vol increases in proportion to square root of time elapsed
- Useful rule-of-thumb even if returns only approximately lognormal
 - But assumes constant return volatility, i.e. flat term structure of volatility
 - At odds with changes in volatility over time and with long-term mean reversion

Simple approaches conditional volatility estimation

Applying the square-root-of-time rule

- Volatility forecast horizon includes trading days, not calendar time
 - Typical year includes about 250-255 trading days
 - Assume 256 trading days, $\sqrt{256} = 16$
 - Annualized volatility pprox 16×daily volatility
- Examples:
 - Long-term average annual volatility of U.S. stock indexes $\approx 16-20$ percent \Rightarrow daily vol $\approx 1-1.25$ percent
 - Swaption normal volatility 80 bps⇒daily vol 5 bps

Volatility forecasting

Simple approaches conditional volatility estimation

Simple conditional volatility estimators

Use moving window incorporating past m trading days' returns **Root mean square:** square root of the sum of squared returns (deviations from zero) divided by the number of observations

$$\sigma_t = \sqrt{\frac{1}{m} \sum_{\tau=1}^m r_{t-m+\tau}^2}$$

Incorporates assumption of zero mean return

Standard deviation: the square root of the sum of squared deviations from the mean return $\bar{r}_t = \frac{1}{m} \sum_{\tau=1}^m r_{t-m+\tau}$ divided by the number of observations minus 1

$$\sigma_t = \sqrt{\frac{1}{m-1}\sum_{\tau=1}^m (r_{t-m+\tau} - \bar{r}_t)^2},$$

• Bias-corrected for 1 degree of freedom lost due to use of \bar{r}_t

GARCH

GARCH model of volatility

- Generalized autoregressive conditionally heteroscedastic model
- Volatility driven by
 - Recent volatility
 - Recent returns
 - Long-term "point of rest" of volatility or "forever vol" $\bar{\sigma}$
- Estimate σ_t made at today's close updates yesterday's estimate σ_{t-1} with latest return r_t
- Look back one period→GARCH(1,1):

$$\sigma_t^2 = \alpha r_t^2 + \beta \sigma_{t-1}^2 + \gamma \bar{\sigma}^2$$

• Feedback to returns via "shock" or "innovation" ϵ_t

$$r_t = \epsilon_t \sigma_{t-1},$$

- Today's return r_t the only pertinent new information on date t
 - ϵ_t assumed i.i.d. with mean 0 and variance 1
 - ϵ_t together with current volatility σ_{t-1} determines new return r_t
 - r_t random but not "free," set by current vol and random shock ϵ_t
- The weights satisfy $\alpha,\beta,\gamma>$ 0 and $\alpha+\beta+\gamma=1$

GARCH

Role of parameters in the GARCH model

- Impact of α : high $\alpha \Rightarrow$
 - Large r_t causes large, immediate change in estimated return volatility σ_t
 - Wider range of variation of σ_t over time
- Impact of β : high $\beta \Rightarrow$
 - σ_t and deviations from $\bar{\sigma}^2$ very persistent
 - Less variation of σ_t over time
- Long-term variance $ar{\sigma}^2 > 0$
 - Presence of $\bar{\sigma}^2$ generates a term structure of volatility
 - Example: $\bar{\sigma}^2$ approximately 1.0–1.15 percent for U.S. equity market (at daily rate)
- Low $\gamma \Rightarrow$ little mean reversion
- Estimates of eta generally not very far from 1, lpha+eta quite close to 1
- Estimated parameter values lead to (hopefully realistic) behavior of volatility

GARCH

Estimating GARCH(1,1) model parameters

- Maximum likelihood method a standard approach
 - Assume conditional normality, shocks ε_t normally distributed, a stronger assumption than i.i.d.:

$$\epsilon_t \sim \mathcal{N}(0, 1) \quad \forall t$$

Joint normal density of *m* return observations⇒log likelihood function

$$\sum_{\tau=1}^{m} \left[-\ln(\sigma_{t-\tau}^2) - \frac{r_t^2}{\sigma_{t-\tau}^2} \right],$$

with initial volatility value σ_0

- Use numerical search procedure to find parameters that maximize log likelihood function
 - Numerical search procedure can be sensitive to initial trial guess
 - $\omega \equiv \gamma \bar{\sigma}^2$ treated as a single parameter
 - γ can then be recovered as 1-lpha-eta and

$$\bar{\sigma} = \sqrt{\frac{\omega}{1 - \alpha - \beta}}$$

GARCH

Influence of past returns in GARCH model

• GARCH(1,1) formula can be recast in terms of most recent and past squared returns (setting *m* = *t*):

$$\begin{split} \sigma_1^2 &= \alpha r_1^2 + \beta \sigma_0^2 + \omega \\ \sigma_2^2 &= \alpha r_2^2 + \beta \sigma_1^2 + \omega = \alpha r_2^2 + \beta (\alpha r_1^2 + \beta \sigma_0^2 + \omega) + \omega \\ &= \alpha r_2^2 + \alpha \beta r_1^2 + \beta^2 \sigma_0^2 + (1 + \beta) \omega \\ \vdots \\ \sigma_t^2 &= \alpha \Sigma_{\tau=1}^t \beta^{t-\tau} r_{\tau}^2 + \Sigma_{\tau=1}^t \beta^{t-\tau} \omega + \beta^t \sigma_0^2 \\ &\approx \alpha \Sigma_{\tau=1}^t \beta^{t-\tau} r_{\tau}^2 + \frac{1}{1-\beta} \omega \end{split}$$

- $\alpha < 1, \beta < 1 \Rightarrow$ small influence of more remote past returns, starting value σ_0
- Tradeoff between influence of long-term variance and that of most recent volatility estimate

GARCH

Example of GARCH(1,1) model estimation

- Applied to S&P 500 index, using m + 1 = 3651 closing-price observations 30Jun2005 to 31Dec2019
- r_1^2 used as starting value σ_0
 - Can also use sample variance of entire time series
- For each pass of the search procedure, successively apply GARCH(1,1) formula to calculate trial values σ₁, σ₂,..., σ_t:

Parameter estimates							
α	0.12195	β	0.85609				
ω	2.40805×10^{-6}	$\bar{\sigma}$ (%)	1.04715				
γ	0.02196						

• Practical application: (re-)estimate parameters infrequently, but use estimated model regularly to forecast volatility

The exponentially-weighted moving average model

Exponentially-weighted moving average model

- Exponentially-weighted moving average (EWMA)
 - A.k.a. RiskMetrics model
 - Variance a weighted average of past returns
 - Weights smaller for more-remote past returns
- Single parameter: decay factor λ
 - Low λ: rapid adaptation to recent returns
 - High λ : slow adaptation to recent returns
- Some limitations of EWMA:
 - EWMA implies a flat term structure of volatility
 - Volatility follows square-root-of-time rule
 - Volatility behaves as a random walk, subject to shocks
 - EWMA can't incorporate recurrent changes in volatility, e.g. earnings season

Volatility forecasting

The exponentially-weighted moving average model

Estimating volatility with the EWMA model

- Typically, assume a value for parameter λ rather than estimate it, and apply a formula
- Current volatility estimate σ_t uses m most recent observed returns r_{t-m+1}, \ldots, r_t
 - Treat λ as known parameter
 - Weight on each squared return $\frac{1-\lambda}{1-\lambda^m}\lambda^{m- au}, au=1,\ldots,m$
 - Apply $\lambda^{m-m} = 1$ for $\tau = m$, most recent (time t) return
 - Apply $\lambda^{m-1} \approx 0$ for $\tau = 1$, most remote (time t m + 1) return

$$\sigma_t^2 = \frac{1-\lambda}{1-\lambda^m} \sum_{\tau=1}^m \lambda^{m-\tau} r_{t-m+\tau}^2$$

•
$$1 - \lambda^m \approx 1 \Rightarrow$$

 $\sigma_i^2 \approx (1)$

$$\sigma_t^2 \approx (1-\lambda) \sum_{\tau=1}^m \lambda^{m-\tau} r_{t-m+\tau}^2$$

- *m* doesn't have to be large
 - mpprox 100 more than adequate unless λ quite close to 1

Volatility forecasting

The exponentially-weighted moving average model

The EWMA model weighting scheme



days since return observation

The graph displays the values of the last 100 of m = 250 EWMA weights $\frac{1-\lambda}{1-\lambda^m}\lambda^{m-\tau}$ for $\lambda = 0.94$ and $\lambda = 0.97$.

The exponentially-weighted moving average model

Impact of the decay factor

- Low λ ⇒recent observations have greater weight:
 - Volatility changes rapidly
 - Recent observations have most information useful for short-term conditional volatility forecasting
- Low λ effectively shortens historical sample size compared to high λ
- Estimates using low λ much more variable than those using high λ
- Estimates using low λ respond more rapidly to new information
- Estimates using low λ may move in the opposite direction from those using high λ
 - Estimates using low λ decline after a sequence of high-magnitude returns, while those using high λ still rising in response

The exponentially-weighted moving average model

Choosing the decay factor

- No agreed method for estimating λ
- Decay factor estimation: λ that minimizes forecast errors, e.g. RMS criterion
- Decay factor may also be chosen judgmentally
- Widely adopted standard settings for decay factor:
 - $\lambda = 0.94$ for short-term (e.g. one-day) forecasts
 - $\lambda = 0.97$ for medium-term (e.g. one-month) forecasts
 - Minimizes RMS of forecast errors for range of assets in original 1994 RiskMetrics study

—The exponentially-weighted moving average model

Effect of the decay factor on the volatility forecast



EWMA estimates of the volatility of daily S&P 500 index returns 01Jul2005 to 31Dec2019, at a daily rate in percent, using decay factors of $\lambda = 0.94$ and $\lambda = 0.99$. Points represent the absolute value of daily return observations.

The exponentially-weighted moving average model

Estimating volatility with the EWMA model

0 21Jul2014 1973.63 NA NA 1 22Jul2014 1983.53 0.00500 0.00000 0.00000 2 23Jul2014 1987.01 0.00175 0.00000 0.00000 3 24Jul2014 1987.98 0.00049 0.00000 0.00000 4 25Jul2014 1978.34 -0.00486 0.00000 0.00000 173 27Mar2015 2061.02 0.00237 0.00051 0.00286 174 30Mar2015 2066.24 0.01216 0.00054 0.08052 175 31Mar2015 2057.89 -0.00397 0.00058 0.04526 176 01Apr2015 2059.69 -0.00397 0.00066 0.00814 177 02Apr2015 2066.96 0.00352 0.00066 0.00844	NA 0×10^{-6} 0×10^{-6} 0×10^{-6} 0×10^{-6}
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177 02Apr2015 2066.96 0.00352 0.00066 0.00814	3×10^{-6}
	4×10^{-6}
	:
246 13Jul2015 2099.60 0.01101 0.04684 5.67368	3×10^{-6}
247 14Jul2015 2108.95 0.00444 0.04984 0.98393	1×10^{-6}
248 15Jul2015 2107.40 -0.00074 0.05302 0.02866	5×10^{-6}
249 16Jul2015 2124.29 0.00798 0.05640 3.59398	3×10^{-6}
250 17Jul2015 2126.64 0.00111 0.06000 0.0733	- 10-6

Return vol of the S&P 500 index, estimated after the close on 17Jul2015 (date t), with m = 250, $\lambda = 0.94$. Return (4th column) expressed as a decimal. Add the 250 values in the last column to get the estimated variance σ_t^2 .

Volatility forecasting

The exponentially-weighted moving average model

Recursive formula for EWMA volatility estimates

• Recursive formula updates most recent volatility estimate with new data on return magnitude

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda)r_t^2$$

- $\rightarrow \text{Easy}$ computation technique, very close to result of full EWMA weighting scheme
- Shows similarity of EWMA to "one-parameter" GARCH
 - But with long-term volatility term $\gamma = 0, \alpha + \beta = 1$
 - λ analogous to eta, $1-\lambda$ analogous to lpha
 - Shocks to volatility permanent, no long-term "forever" vol
 - Also known as integrated GARCH or IGARCH(1,1)
- EWMA estimate usually close to unrestricted GARCH(1,1) estimate
- "Starter value" (orange in example on next slide):
 - Root mean square, using first month (21 days) of return of observations
 - Starter value method not crucial, converges quickly (esp. for low λ)

Volatility behavior and forecasting

Volatility forecasting

The exponentially-weighted moving average model

Recursive formula for EWMA volatility estimates

t	Date	S_t	r _t (%)	$\lambda \sigma_{t-1}^2$	$(1-\lambda)r_t^2$	σ_t (%)
1	30Jun2005	1191.33	NA	NA	NA	0.55583
2	01Jul2005	1194.44	0.2607	0.29041×10^{-4}	0.40783×10^{-6}	0.54267
3	05Jul2005	1204.99	0.8794	0.27682×10^{-4}	4.63987×10^{-6}	0.56853
4	06Jul2005	1194.94	-0.8375	0.30383×10^{-4}	4.20873×10 ⁻⁶	0.58815
5	07Jul2005	1197.87	0.2449	0.32516×10^{-4}	0.35986×10^{-6}	0.57338
6	08Jul2005	1211.86	1.1611	0.30903×10^{-4}	8.08946×10^{-6}	0.62444
7	11Jul2005	1219.44	0.6235	0.36653×10^{-4}	2.33279×10^{-6}	0.62439
						-
:	:	:	:	:	:	:
3644	19Dec2019	3205.37	0.4449	0.24206×10^{-4}	1.18778×10^{-6}	0.50392
3645	20Dec2019	3221.22	0.4933	0.23870×10^{-4}	1.45986×10^{-6}	0.50329
3646	23Dec2019	3224.01	0.0866	0.23810×10^{-4}	0.04497×10^{-6}	0.48842
3647	24Dec2019	3223.38	-0.0195	0.22424×10^{-4}	0.00229×10^{-6}	0.47356
3648	26Dec2019	3239.91	0.5115	0.21080×10^{-4}	1.56983×10^{-6}	0.47592
3649	27Dec2019	3240.02	0.0034	0.21291×10^{-4}	0.00007×10^{-6}	0.46142
3650	30Dec2019	3221.29	-0.5798	0.20014×10^{-4}	2.01673×10^{-6}	0.46937
3651	31Dec2019	3230.78	0.2942	0.20709×10^{-4}	0.51921×10^{-6}	0.46074

Return vol of the S&P 500 index, estimated daily using the recursive formula, with $\lambda = 0.94$. Initial vol estimate: RMS of the 20 daily returns 01Jul2005–29Jul2005.

- Volatility forecasting
 - The exponentially-weighted moving average model

GARCH(1,1) and EWMA volatility estimates



Daily estimates of S&P 500 index's annualized return volatility, 30Jun2005 to 31Dec2019. EWMA estimates with $\lambda=0.94$ GARCH(1,1) estimates use parameters $\alpha=0.12195, \beta=0.85609, \gamma\bar{\sigma}^2=2.40805\times 10^{-6}$. The annualized realized return volatility was 15.69 percent over the period.